MOMENTS EQUALITIES FOR NONNEGATIVE INTEGER-VALUED RANDOM VARIABLES

MOHAMED I RIFFI
ASSOCIATE PROFESSOR OF MATHEMATICS
DEPARTMENT OF MATHEMATICS
ISLAMIC UNIVERSITY OF GAZA
GAZA, PALESTINE

ABSTRACT. We present and prove two theorems about equalities for the nth moment of nonnegative integer-valued random variables. These equalities generalize the well known equality for the first moment of a nonnegative integer-valued random variable, X, in terms of its cumulative distribution function, or in terms of $P(X > x)$.

1. INTRODUCTION

There is a well-known equality for the nth moment of a nonnegative random variable, Y, in terms of $P(Y > y)$ as in [3, p. 33], for example. A similar equality for the first moment of a nonnegative integer-valued random variable is also well known and used a lot in the literature (See [1, p. 43], for example). In this paper we are going to prove in this paper is a generalization of these equalities in the discrete case.

In the next section we will prove a generalization of the well known equality in the discrete case. Our equality gives a neat formula of the nth moment, when it exists, for nonnegative integer-valued variables.

2. MAIN THEOREMS

In this section we prove two identities that will be used to prove our main theorems. The first identity is used to express a product in terms of the form $(X - i)$ as a finite sum of products of similar terms when the sum ranges from 1 to a nonnegative integer x. The second identity is used to express x^n as a finite sum that ranges from 1 to a nonnegative integer x.

Date: September 3, 2002.
2000 *Mathematics Subject Classification.* Primary 60A99. Secondary 62B99.
Key words and phrases. Expectation, Moments, Equalities.

Before we proceed to the main theorems, we need the following lemma.

Lemma 2.1.
$$\binom{x}{n+1} = \sum_{j=1}^{x-n} \binom{x-j}{n}.$$

Proof. Remember Pascal's identity, namely
$$\binom{x}{n+1} = \binom{x-1}{n} + \binom{x-1}{n+1}$$
Apply this identity to each last term on the right-hand side of the resulting equation. Continue this procedure $x-n-1$ times to get
$$\begin{aligned}\binom{x}{n+1} &= \sum_{j=1}^{x-n-1} \binom{x-j}{n} + \binom{n+1}{n+1}\\ &= \sum_{j=1}^{x-n} \binom{x-j}{n}.\end{aligned}$$
□

Lemma 2.2. *Let* $g_n(x) := \sum_{i=1}^{x} \prod_{j=1}^{n} (i-j)$ *and* $f_n(x) := \prod_{j=0}^{n} (x-j)$. *Then* $(n+1)g_n(x) = f_n(x) = (x)_{n+1}$, *where* $(z)_m = z(z-1)\dots(z-m+1)$.

Proof. Notice that
$$f_n(x) = x(x-1)\cdots(x-n) = (x)_{n+1} = (n+1)!\binom{x}{n+1}.$$

We also notice that $g_n(x)$ is a finite sum of terms of the form $(x-j)(x-j-1)(x-j-2)\cdots$ of length n. Such a general term can be expressed as $(x-j)_n$ for $j = 1, 2, \dots, n$. The number of these terms is $x-n$. Therefore,
$$g_n(x) = \sum_{j=1}^{x-n} (x-j)_n. \tag{2.1}$$
Since $(x-j)_n = n!\binom{x-j}{n}$, (2.1) can be written as
$$g_n(x) = n! \sum_{j=1}^{x-n} \binom{x-j}{n}.$$
By Lemma 2.1, we have
$$g_n(x) = n!\binom{x}{n+1}.$$

This implies that $(n+1)g_n(x) = f_n(x)$. $\square$

Remark 2.1. *Lemma 2.2 simply says that for $x \geq n \geq 2$,*

$$\prod_{i=0}^{n-1}(x-i) = n\sum_{y=1}^{x}\prod_{i=1}^{n-1}(y-i). \tag{2.2}$$

Theorem 2.1. *Let X be a nonnegative integer-valued random variable and $n \geq 2$. Then*

$$E\left(\prod_{i=0}^{n-1}(X-i)\right) = n\sum_{x=n-1}^{\infty}\left(\prod_{i=0}^{n-2}(x-i)P(X>x)\right), \tag{2.3}$$

provided that the sum on the right-hand side of (2.3) *exists.*

Proof. The proof of this theorem mainly depends on (2.2) which is proved in Lemma 2.2.

$$\begin{aligned}
\mathrm{E}\left(\prod_{i=0}^{n-1}(X-i)\right) &= \sum_{x\geq 0}\prod_{i=0}^{n-1}(x-i)P(X=x) \\
&= n\sum_{x\geq 0}P(X=x)\sum_{y=1}^{x}\prod_{i=1}^{n-1}(y-i) \qquad \text{(by (2.2))} \\
&= n\sum_{y=1}^{\infty}\sum_{x=y}^{\infty}\prod_{i=1}^{n-1}(y-i)P(X=x) \text{ (Fubini's Theorem)} \\
&= n\sum_{y=1}^{\infty}\prod_{i=1}^{n-1}(y-i)P(X\geq y) \\
&= n\sum_{x=n}^{\infty}\prod_{i=1}^{n-1}(x-i)P(X\geq x) \\
&= n\sum_{x=n-1}^{\infty}\prod_{i=0}^{n-2}(x-i)P(X>x),
\end{aligned} \tag{2.4}$$

where we have changed the dummy variable y to x. $\square$

Remark 2.2. *We may start the sum on the right-hand side of* (2.3) *at $x=0$, since the first $n-1$ terms of the product $n\prod_{i=0}^{n-2}(x-i)$ are equal to 0.*

Remark 2.3. *The case $n = 1$ is well known and can be stated separately for notational convenience, where the proof can be found in [1].*

$$E(X) = \sum_{x=0}^{\infty} P(X > x).$$

Theorem 2.2 (Riffi's Equality)**.** *Let X be a nonnegative integer-valued random variable and $n \geq 1$. Then*

$$\begin{aligned} E(X^n) &= \sum_{x=0}^{\infty} \sum_{i=1}^{n} \binom{n}{i} x^{n-i} P(X > x) \\ &= \sum_{x=0}^{\infty} [(1+x)^n - x^n] P(X > x), \qquad (2.5) \end{aligned}$$

provided that the sum on the right-hand sides of (2.5) *exists.*

Proof. We proceed by induction on n. By Remark 2.3 the equality holds for $n = 1$. Assume that it holds for all m such that $m \leq n-1$. Then we show it holds for $m = n$. By assumption,

$$\mathrm{E}(X^m) = \sum_{x=0}^{\infty} l^m(x) P(X > x), \quad m \leq n-1, \qquad (2.6)$$

where $l^n(x) = (1+x)^n - x^n$ for $n \geq 1$. By Theorem 2.1 and Remark 2.2 we have

$$\mathrm{E}(X^n) = \mathrm{E}(\phi_n(X)) + \sum_{x=0}^{\infty} \psi_n(x) P(X > x),$$

where $\psi_n(x) = n \prod_{i=0}^{n-2}(x-i)$ and $\phi_n(x) = x^n - \prod_{i=0}^{n-1}(x-i)$.

The polynomial $\psi_n(x)$ is of the form

$$\psi_n(x) = \sum_{i=1}^{n-1} b_{n,i} x^i, \qquad (2.7)$$

where the coefficients $b_{n,i}$'s are obtained for $1 \leq i \leq n-2$ and $2 \leq n$ by

$$b_{n,i} = \sum_{j=0}^{n-i-1} d_{n,j} \binom{n-j}{n-i-j}, \qquad (2.8)$$

The coefficients $d_{n,i}, 1 \leq i \leq n-2$ are given by

$$d_{n,i} = (-1)^i \sum_{\substack{j_1, j_2, \ldots, j_i = 1 \\ j_1 < j_2 < \cdots < j_i}}^{n-1} \prod_{k=1}^{i} (n - j_k). \qquad (2.9)$$

and $d_{n,0} = 1$.

The polynomial $\phi_n(x)$ is of the form $\phi_n(x) = \sum_{i=1}^{n-1} e_i x^i$, where the e_i's are integers. Hence, to find the expectation of $\phi_n(x)$, we apply the (2.6) to each of its terms. Therefore,

$$\mathrm{E}(\phi_n(X)) = \sum_{x=0}^{\infty} \left(\sum_{i=1}^{n-2} e_{n,i} l^i(x) \right) P(X > x). \tag{2.10}$$

The sum $\sum_{i=1}^{n-2} e_{n,i} l^i(x)$ can be rewritten as $\sum_{i=0}^{n-2} a_{n,i} x^i$. In other words (2.10) becomes

$$\mathrm{E}(\phi_n(X)) = \sum_{x=0}^{\infty} \left(\sum_{i=0}^{n-2} a_{n,i} x^i \right) P(X > x). \tag{2.11}$$

Here the constants $a_{n,i}$ are given by

$$a_{n,i} = \sum_{j=1}^{n-i-1} c_{n,j} \binom{n-j}{n-i-j}, \tag{2.12}$$

where coefficients $c_{n,i}$'s are obtained for $1 \leq i \leq n-1$ and $2 \leq n$ by

$$c_{n,i} = (-1)^{i+1} \sum_{\substack{j_1, j_2, \ldots, j_i = 1 \\ j_1 < j_2 < \cdots < j_i}}^{n-1} \prod_{k=1}^{i} (n - j_k). \tag{2.13}$$

Notice that we have for $1 \leq i \leq n-2$ and $2 \leq n$

$$a_{n,i} + b_{n,i} = \binom{n}{i}. \tag{2.14}$$

To complete the proof, we need (2.14) to prove the following lemma.

Lemma 2.3.

$$\sum_{i=0}^{n-2} a_{n,i} x^i + \sum_{i=1}^{n-1} b_{n,i} x^i = \sum_{i=1}^{n} \binom{n}{i} x^{n-i}. \tag{2.15}$$

Proof.

$$\begin{aligned} \sum_{i=0}^{n-2} a_{n,i} x^i + \sum_{i=1}^{n-1} b_{n,i} x^i =& a_{n,0} + \sum_{i=1}^{n-2} a_{n,i} x^i + \sum_{i=1}^{n-2} b_{n,i} x^i + b_{n,n-1} x^{n-1} \\ =& 1 + \sum_{i=1}^{n-2} \binom{n}{i} x^i + n x^{n-1}, \end{aligned}$$

since $b_{n,n-1} = n$ for all $n \geq 2$.

Note that $\binom{n}{i} = \binom{n}{n-i}$. So, writing the sum $\sum_{i=1}^{n-2}\binom{n}{i}x^i$ in reverse order, we obtain

$$\sum_{i=1}^{n-2}\binom{n}{i}x^i = \sum_{i=1}^{n-2}\binom{n}{n-i}x^i = \sum_{i=2}^{n-1}\binom{n}{i}x^{n-i}, \tag{2.16}$$

and this completes the proof. □

Hence the proof of Theorem 2.2 is complete. □

References

[1] Kai Lai Chung, *A Course in Probability Theory*, Second Edition, Academic Press, New York, 1974.

[2] George Casella, Roger L. Berger, *Statistical Inference*, Duxbury Press, California, 1990.

[3] Richard Durret, *Probability: Theory and Examples*, Wadsworth, California, 1991.

Islamic University of Gaza, P.O.Box 108, Gaza, Palestine
E-mail address: `miriffi@hotmail.com`